

# Pemodelan ARIMA

Dhaniel Ilyas, S.E. M.Sc.

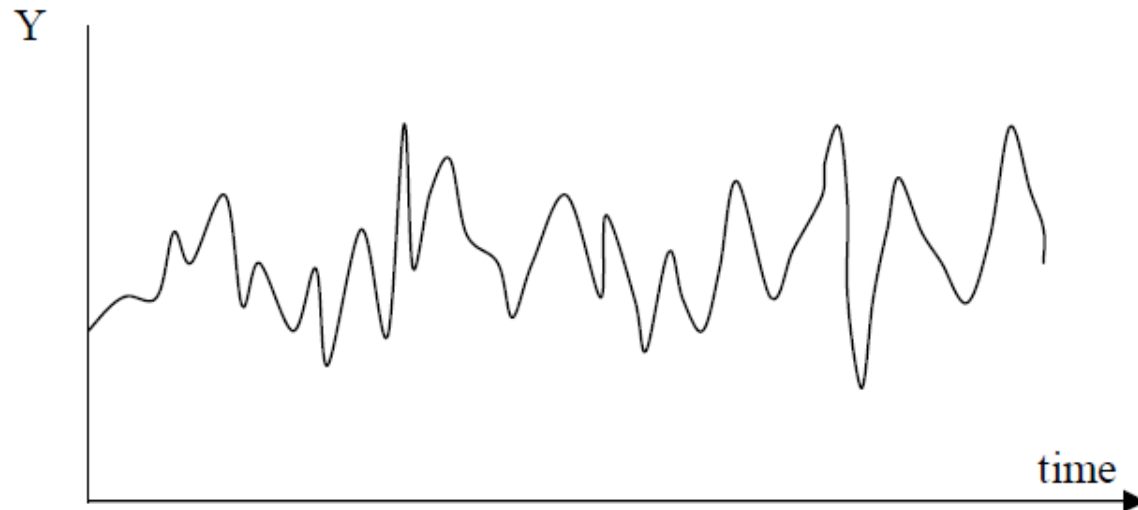
[dhaniel.ilyas@gmail.com](mailto:dhaniel.ilyas@gmail.com)

2017

# Pemahaman Akan Pola Umum Akan Data Seri Waktu

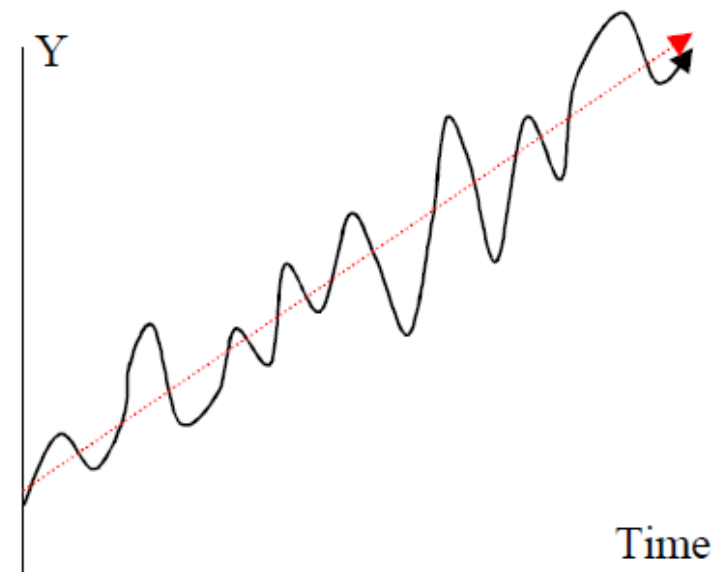
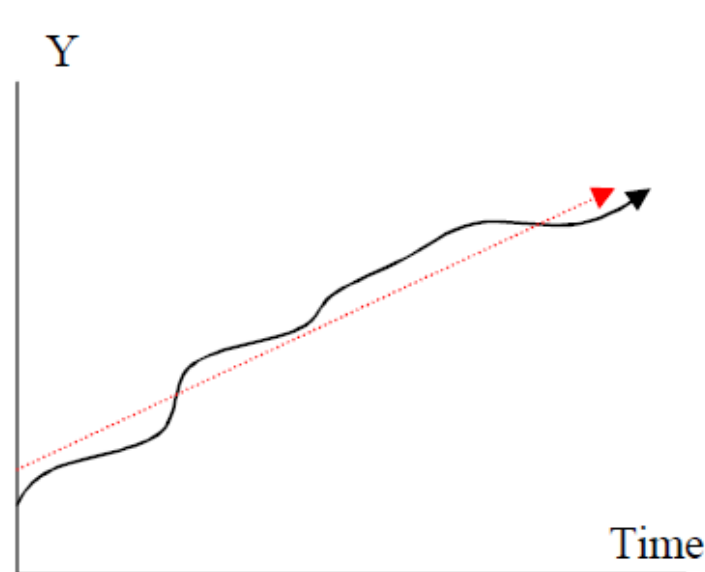
## Random patterns:

Random time series are the result of many influences that act independently to yield nonsystematic and non-repeating patterns about some average value. Purely random series have a constant mean and no systematic patterns. Simply averaging model are often the most best and accurate way to forecast them.

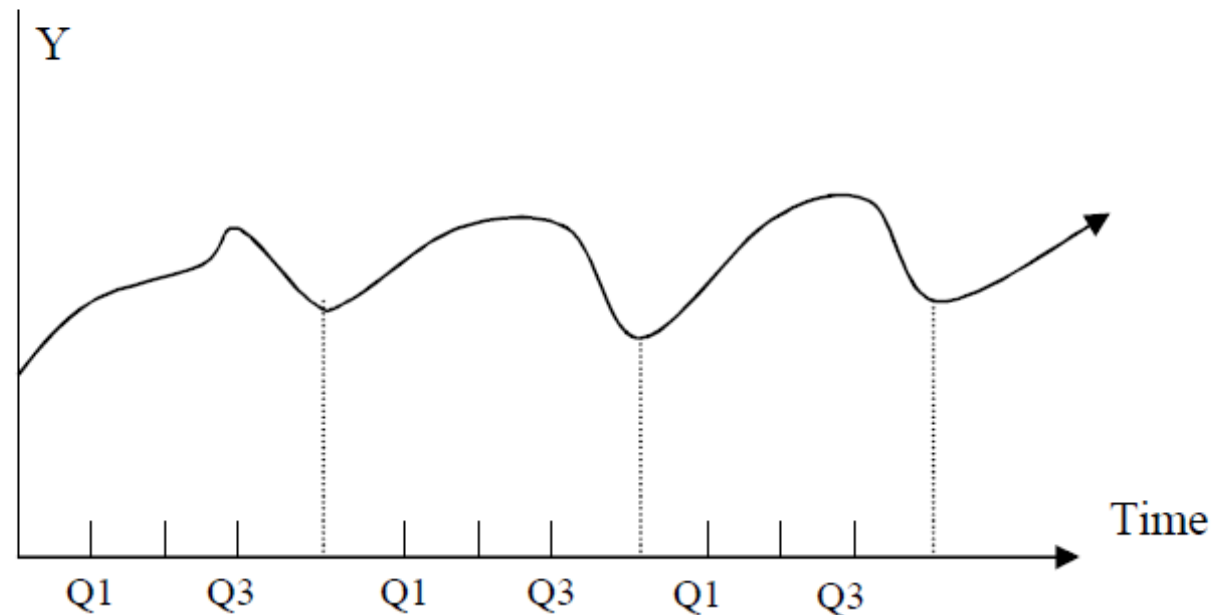


### **Trend patterns:**

A trend is a general increase or decrease in a time series that lasts for approximately seven or more periods. A trend can be a period-to-period increase or decrease that follows as a straight-line, this kind of pattern is called a linear trend. Trends are not necessarily linear because there are a large number of nonlinear causal influences that yield nonlinear series. Trends may cause by long-term factors or population changes, growth during production and technology introductions, changes in economic conditions, and so on.

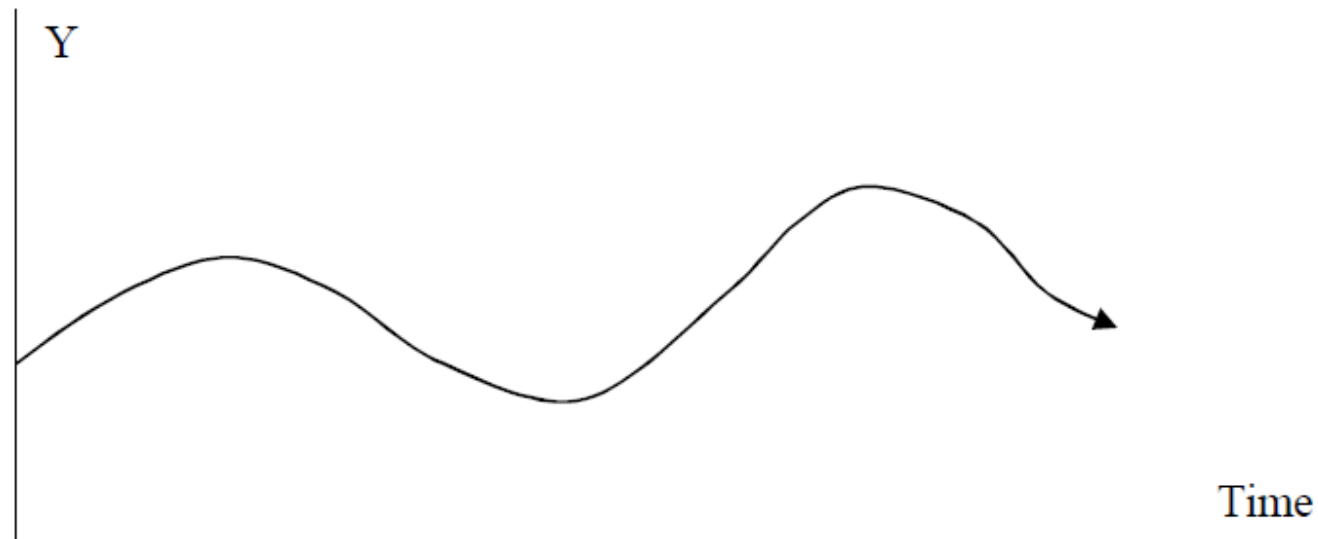


**Seasonal Patterns**: Seasonal series result from events that are periodic and recurrent (e.g., monthly, quarterly, changes recurring each year). Common seasonal influences are climate, human habits, holidays, repeating promotions, and so on.

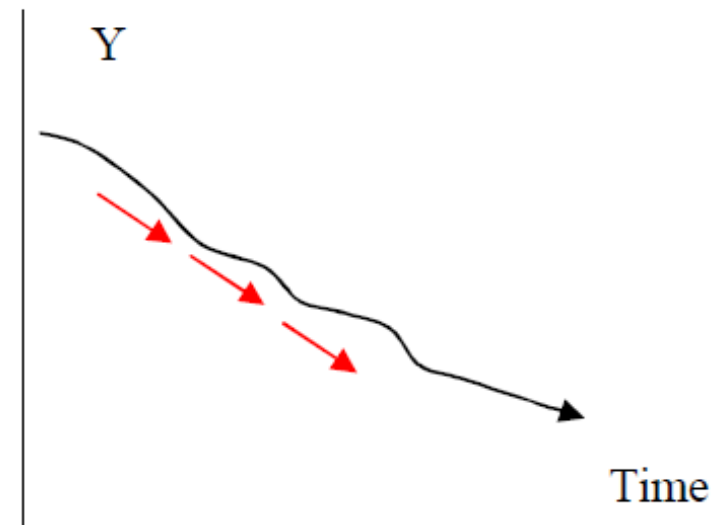
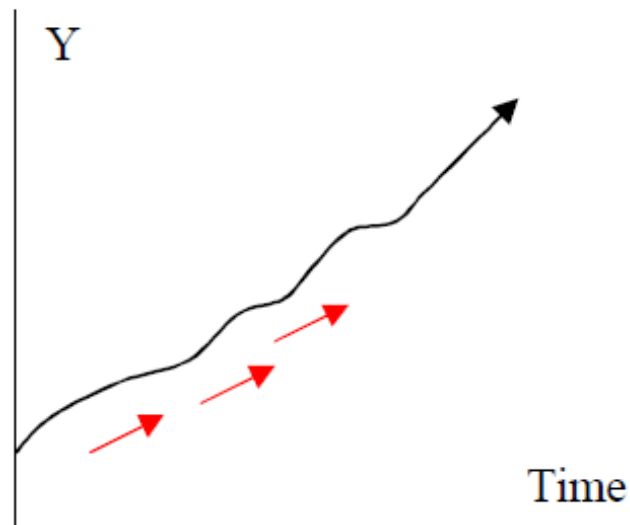


### **Cyclical Patterns:**

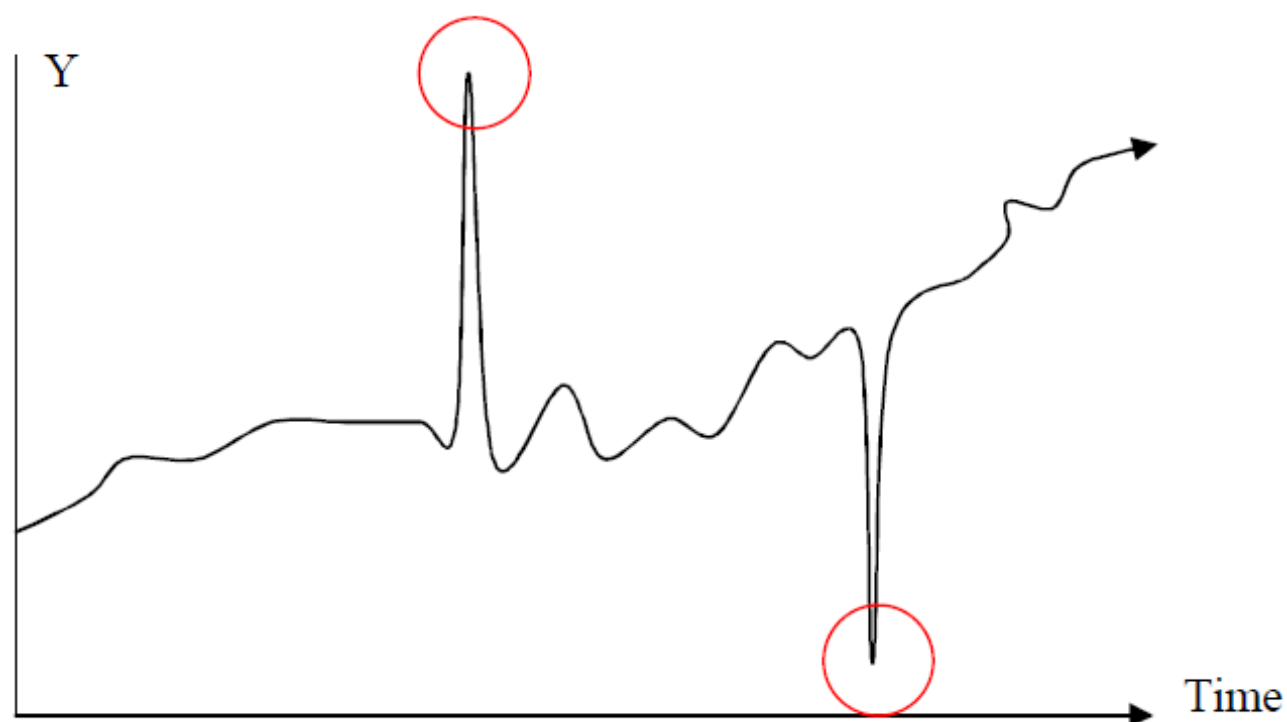
Economic and business expansions and contractions are most frequent cause of cyclical influences on time series. These influences most often last for two to five years and recur, but with no known period. In the search for explanation of cyclical movements, many theories have been proposed, including sunspots, positions of planets, stars, long wave movements in weather conditions, population life cycles, the growth and decay cycle of new products and technologies, product life cycles, and the economic cycles. Cyclical influences are difficult to forecast because they are recurrent but not periodic.



**Autocorrelated Patterns:** Correlation measures the degree of dependence or association between two variables. The Term autocorrelation means that the value of a series in one time period is related to the value of itself in previous periods. With autocorrelation, there is an automatic correlation between observations in a series.



**Outliers:** The analysis of past data can be made very complex when the included values are not typical of the past or future. These non-typical values are called outliers, which are very large or small observations that are not indicative of repeating past or future patterns. Outliers occur because of unusual events.



# Structural Break Test!

- ❖ Lakukan *structural break test* sebelum melakukan *stationarity test*
- ❖ Setelah pasti ada/tidak adanya *structural break* baru gunakan test stasionaritas yang tepat!
- ❖ Perhatikan penjelasan Peter Foldvari akan cara mentesnya. (buka file lec5a.pdf)

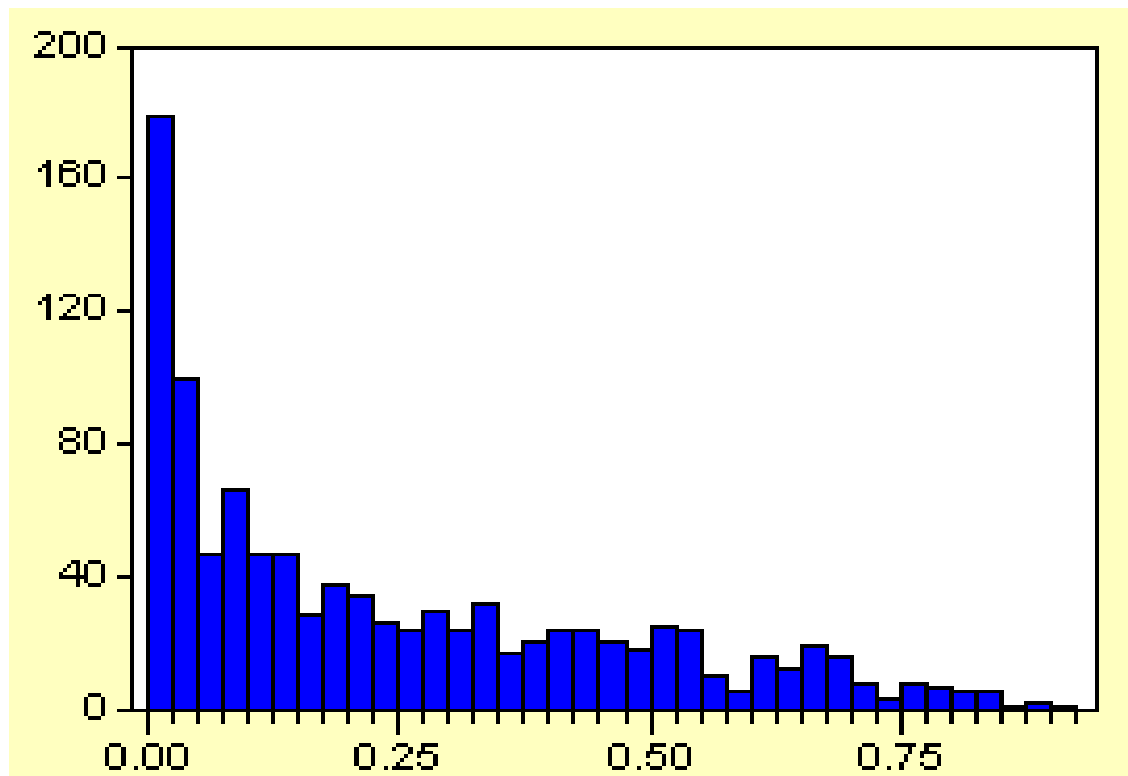


# Stationarity and Unit Root Testing

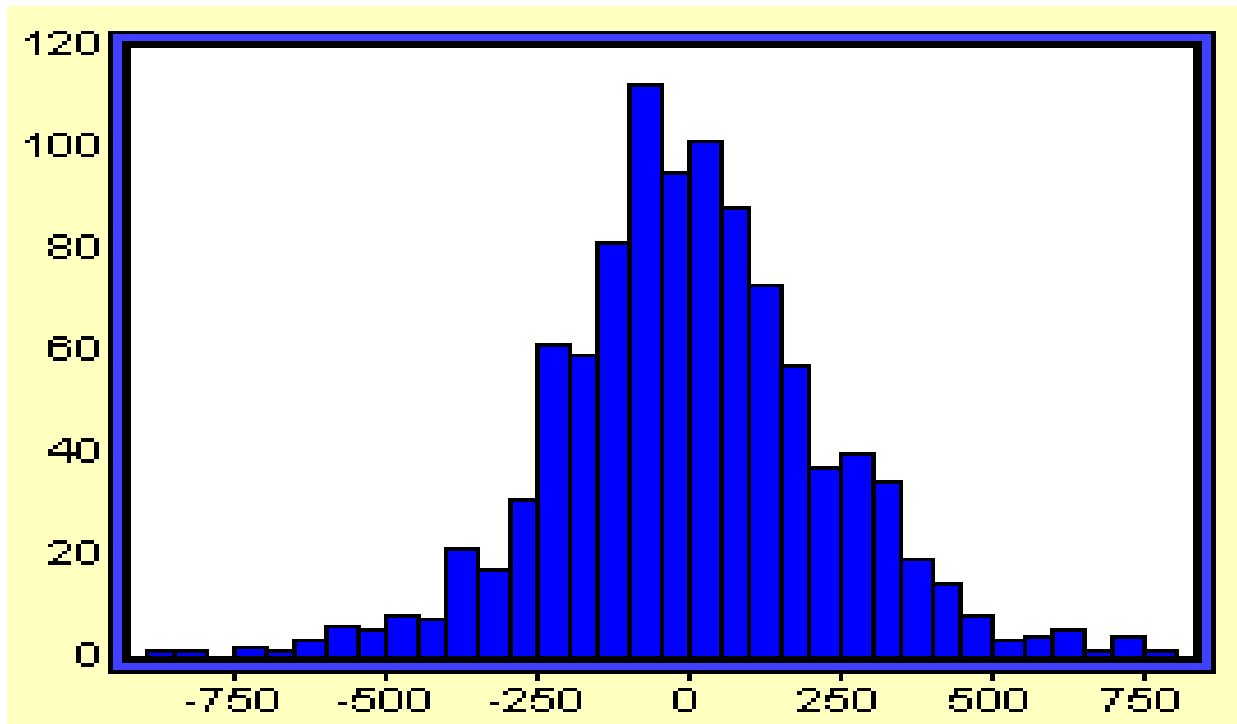
## Why do we need to test for Non-Stationarity?

- ❖ The stationarity or otherwise of a series can strongly influence its behaviour and properties - e.g. persistence of shocks will be infinite for nonstationary series
- ❖ Spurious regressions. If two variables are trending over time, a regression of one on the other could have a high  $R^2$  even if the two are totally unrelated
- ❖ If the variables in the regression model are not stationary, then it can be proved that the standard assumptions for asymptotic analysis will not be valid. In other words, the usual “ $t$ -ratios” will not follow a  $t$ -distribution, so we cannot validly undertake hypothesis tests about the regression parameters.

## Value of $R^2$ for 1000 Sets of Regressions of a Non-stationary Variable on another Independent Non-stationary Variable



## Value of $t$ -ratio on Slope Coefficient for 1000 Sets of Regressions of a Non-stationary Variable on another Independent Non-stationary Variable



## Two types of Non-Stationarity

- ❖ Various definitions of non-stationarity exist
- ❖ In this chapter, we are really referring to the weak form or covariance stationarity
- ❖ There are two models which have been frequently used to characterise non-stationarity: the random walk model with drift:

$$y_t = \mu + y_{t-1} + u_t \quad (1)$$

and the deterministic trend process:

$$y_t = \alpha + \beta t + u_t \quad (2)$$

where  $u_t$  is iid in both cases.

# Stochastic Non-Stationarity

- ❖ Note that the model (1) could be generalised to the case where  $y_t$  is an explosive process:

$$y_t = \mu + \phi y_{t-1} + u_t$$

where  $\phi > 1$ .

- ❖ Typically, the explosive case is ignored and we use  $\phi = 1$  to characterise the non-stationarity because
  - $\phi > 1$  does not describe many data series in economics and finance.
  - $\phi > 1$  has an intuitively unappealing property: shocks to the system are not only persistent through time, they are propagated so that a given shock will have an increasingly large influence.

## Stochastic Non-stationarity: The Impact of Shocks

- ❖ To see this, consider the general case of an AR(1) with no drift:

$$y_t = \phi y_{t-1} + u_t \quad (3)$$

Let  $\phi$  take any value for now.

- ❖ We can write:

$$y_{t-1} = \phi y_{t-2} + u_{t-1}$$

$$y_{t-2} = \phi y_{t-3} + u_{t-2}$$

- ❖ Substituting into (3) yields:

$$\begin{aligned} y_t &= \phi(\phi y_{t-2} + u_{t-1}) + u_t \\ &= \phi^2 y_{t-2} + \phi u_{t-1} + u_t \end{aligned}$$

- ❖ Substituting again for  $y_{t-2}$ :

$$\begin{aligned} y_t &= \phi^2(\phi y_{t-3} + u_{t-2}) + \phi u_{t-1} + u_t \\ &= \phi^3 y_{t-3} + \phi^2 u_{t-2} + \phi u_{t-1} + u_t \end{aligned}$$

- ❖  $T$  successive substitutions of this type lead to:

$$y_t = \phi^T y_0 + \phi u_{t-1} + \phi^2 u_{t-2} + \phi^3 u_{t-3} + \dots + \phi^T u_0 + u_t$$

# The Impact of Shocks for Stationary and Non-stationary Series

❖ We have 3 cases:

1.  $\phi < 1 \Rightarrow \phi^T \rightarrow 0$  as  $T \rightarrow \infty$

So the shocks to the system gradually die away.

2.  $\phi = 1 \Rightarrow \phi^T = 1 \forall T$

So shocks persist in the system and never die away. We obtain:

$$y_t = y_0 + \sum_{t=0}^{\infty} u_t \quad \text{as } T \rightarrow \infty$$

So just an infinite sum of past shocks plus some starting value of  $y_0$ .

3.  $\phi > 1$ . Now given shocks become more influential as time goes on, since if  $\phi > 1$ ,  $\phi^3 > \phi^2 > \phi$  etc.

## Detrending a Stochastically Non-stationary Series

- ❖ Going back to our 2 characterisations of non-stationarity, the r.w. with drift:

$$y_t = \mu + y_{t-1} + u_t \quad (1)$$

and the trend-stationary process

$$y_t = \alpha + \beta t + u_t \quad (2)$$

- ❖ The two will require different treatments to induce stationarity. The second case is known as deterministic non-stationarity and what is required is detrending.
- ❖ The first case is known as stochastic non-stationarity. If we let

$$\Delta y_t = y_t - y_{t-1}$$

and

$$L y_t = y_{t-1}$$

so

$$(1-L) y_t = y_t - L y_t = y_t - y_{t-1}$$

If we take (1) and subtract  $y_{t-1}$  from both sides:

$$y_t - y_{t-1} = \mu + u_t$$

$$\Delta y_t = \mu + u_t$$

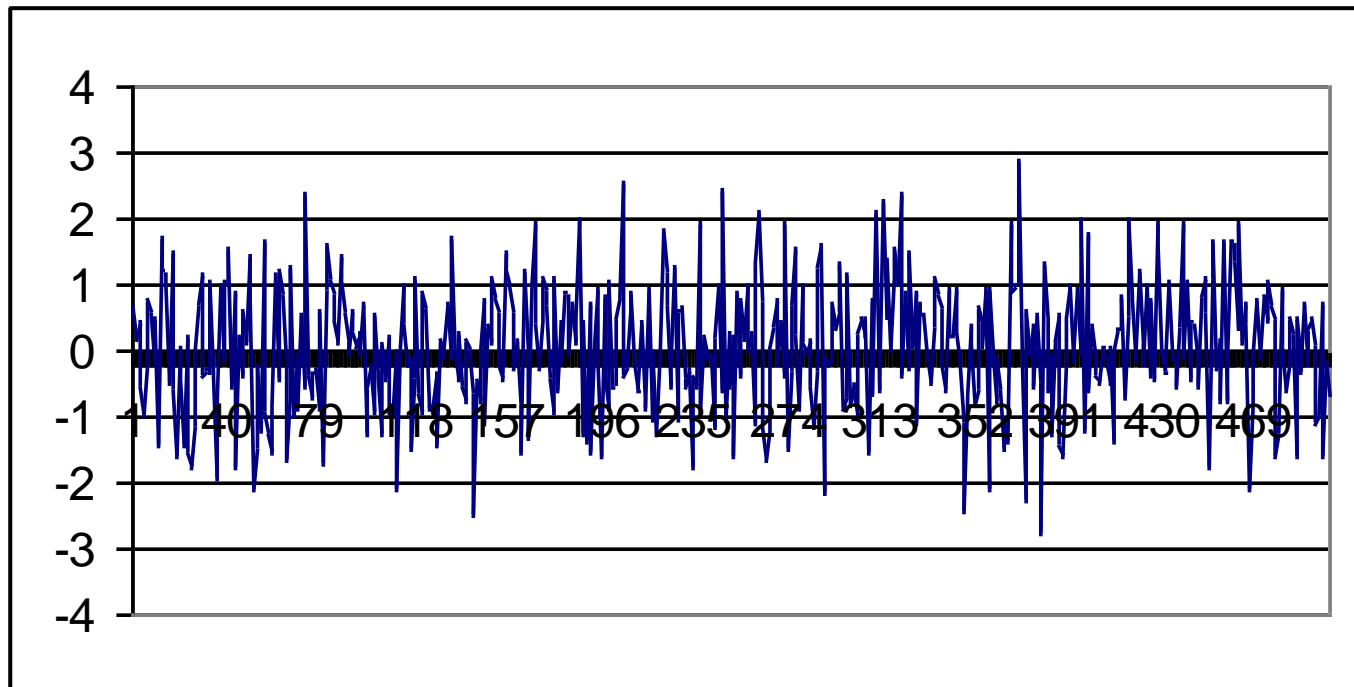
We say that we have induced stationarity by “differencing once”.



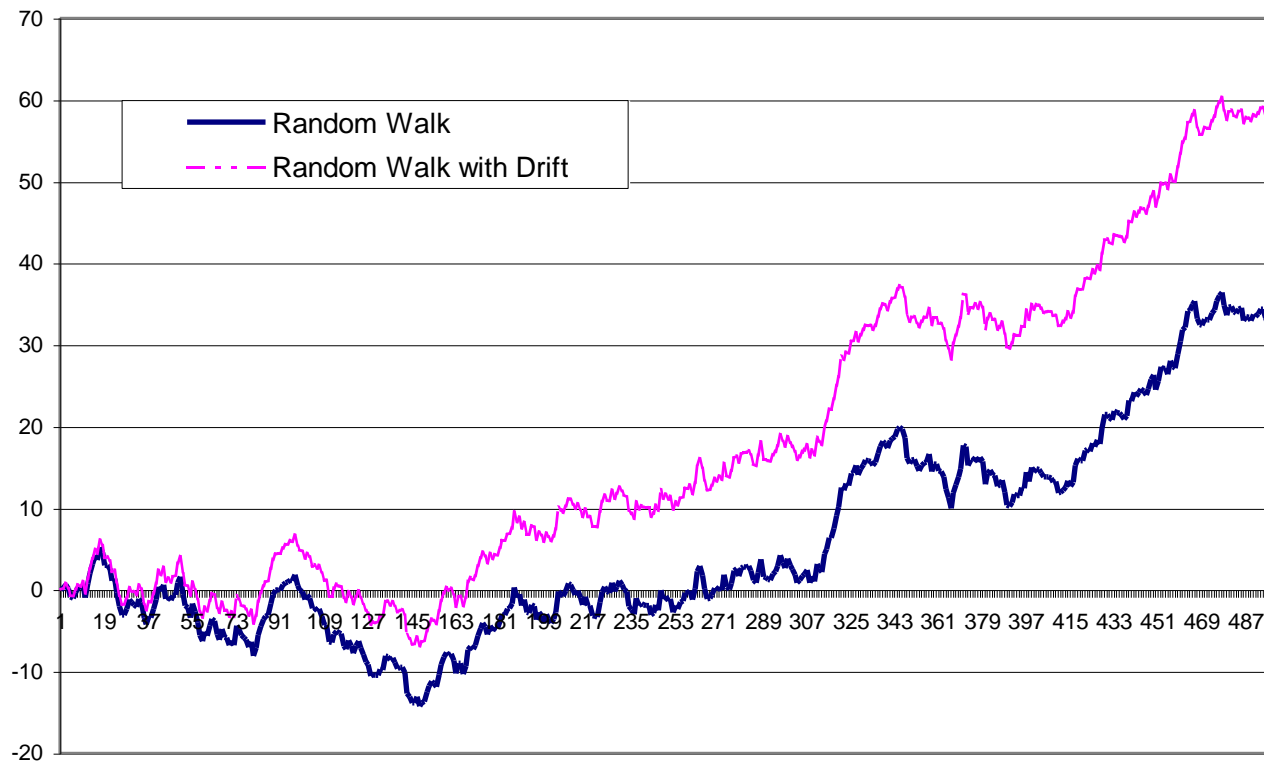
## Detrending a Series: Using the Right Method

- ❖ Although trend-stationary and difference-stationary series are both “trending” over time, the correct approach needs to be used in each case.
- ❖ If we first difference the trend-stationary series, it would “remove” the non-stationarity, but at the expense of introducing an MA(1) structure into the errors.
- ❖ Conversely if we try to detrend a series which has stochastic trend, then we will not remove the non-stationarity.
- ❖ We will now concentrate on the stochastic non-stationarity model since deterministic non-stationarity does not adequately describe most series in economics or finance.

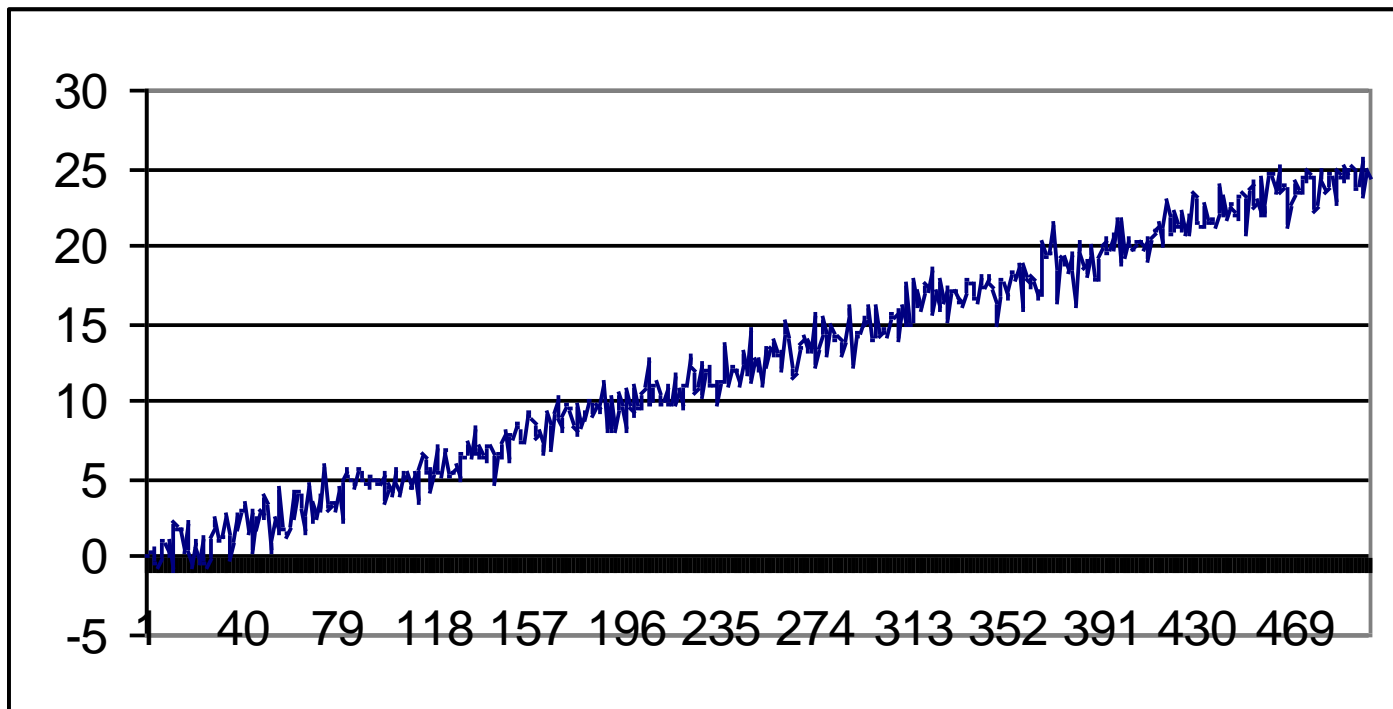
## Sample Plots for various Stochastic Processes: A White Noise Process



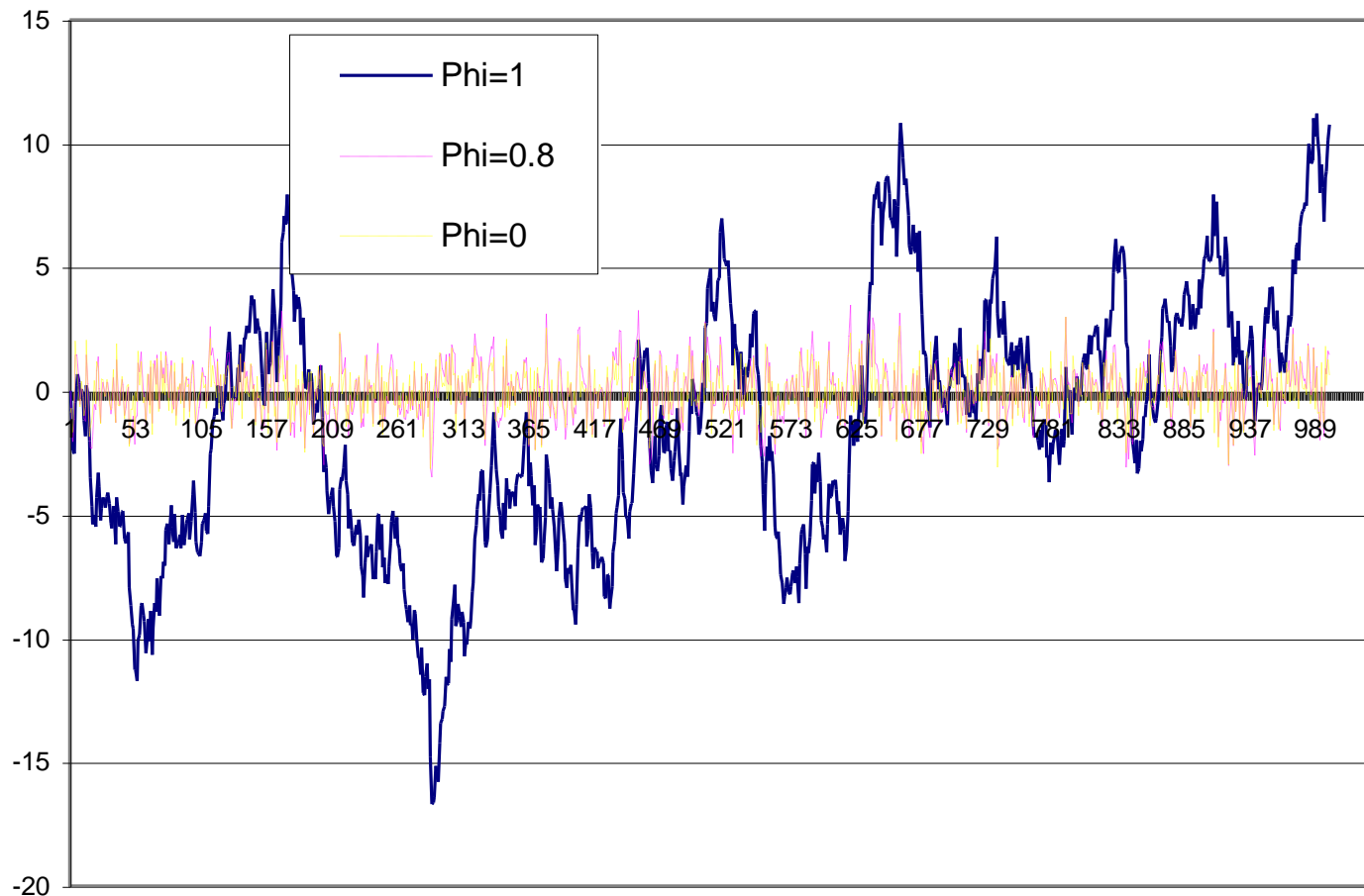
## Sample Plots for various Stochastic Processes: A Random Walk and a Random Walk with Drift



## Sample Plots for various Stochastic Processes: A Deterministic Trend Process



## Autoregressive Processes with differing values of $\phi$ (0, 0.8, 1)



# Definition of Non-Stationarity

- ❖ Consider again the simplest stochastic trend model:

$$y_t = y_{t-1} + u_t$$

or

$$\Delta y_t = u_t$$

- ❖ We can generalise this concept to consider the case where the series contains more than one “unit root”. That is, we would need to apply the first difference operator,  $\Delta$ , more than once to induce stationarity.

## Definition

If a non-stationary series,  $y_t$  must be differenced  $d$  times before it becomes stationary, then it is said to be integrated of order  $d$ . We write  $y_t \sim I(d)$ .

So if  $y_t \sim I(d)$  then  $\Delta^d y_t \sim I(0)$ .

An  $I(0)$  series is a stationary series

An  $I(1)$  series contains one unit root,

$$\text{e.g. } y_t = y_{t-1} + u_t$$

## Characteristics of $I(0)$ , $I(1)$ and $I(2)$ Series

- ❖ An  $I(2)$  series contains two unit roots and so would require differencing twice to induce stationarity.
- ❖  $I(1)$  and  $I(2)$  series can wander a long way from their mean value and cross this mean value rarely.
- ❖  $I(0)$  series should cross the mean frequently.
- ❖ The majority of economic and financial series contain a single unit root, although some are stationary and consumer prices have been argued to have 2 unit roots.

## How do we test for a unit root?

- ❖ The early and pioneering work on testing for a unit root in time series was done by Dickey and Fuller (Dickey and Fuller 1979, Fuller 1976). The basic objective of the test is to test the null hypothesis that  $\phi=1$  in:

$$y_t = \phi y_{t-1} + u_t$$

against the one-sided alternative  $\phi < 1$ . So we have

$H_0$ : series contains a unit root

vs.  $H_1$ : series is stationary.

- ❖ We usually use the regression:

$$\Delta y_t = \psi y_{t-1} + u_t$$

so that a test of  $\phi=1$  is equivalent to a test of  $\psi=0$  (since  $\phi-1=\psi$ ).



## Different forms for the DF Test Regressions

- ❖ Dickey Fuller tests are also known as  $\tau$  tests:  $\tau$ ,  $\tau_\mu$ ,  $\tau_\tau$ .
- ❖ The null ( $H_0$ ) and alternative ( $H_1$ ) models in each case are

i)  $H_0: y_t = y_{t-1} + u_t$   
 $H_1: y_t = \phi y_{t-1} + u_t, \phi < 1$

This is a test for a random walk against a stationary autoregressive process of order one (AR(1))

ii)  $H_0: y_t = y_{t-1} + u_t$   
 $H_1: y_t = \phi y_{t-1} + \mu + u_t, \phi < 1$

This is a test for a random walk against a stationary AR(1) with drift.

iii)  $H_0: y_t = y_{t-1} + u_t$   
 $H_1: y_t = \phi y_{t-1} + \mu + \lambda t + u_t, \phi < 1$

This is a test for a random walk against a stationary AR(1) with drift and a time trend.

# Computing the DF Test Statistic

- ❖ We can write

$$\Delta y_t = u_t$$

where  $\Delta y_t = y_t - y_{t-1}$ , and the alternatives may be expressed as

$$\Delta y_t = \psi y_{t-1} + \mu + \lambda t + u_t$$

with  $\mu = \lambda = 0$  in case i), and  $\lambda = 0$  in case ii) and  $\psi = \phi - 1$ . In each case, the tests are based on the  $t$ -ratio on the  $y_{t-1}$  term in the estimated regression of  $\Delta y_t$  on  $y_{t-1}$ , plus a constant in case ii) and a constant and trend in case iii). The test statistics are defined as

$$\text{test statistic} = \frac{\hat{\psi}}{\hat{\sigma}_\psi}$$

- ❖ The test statistic does not follow the usual  $t$ -distribution under the null, since the null is one of non-stationarity, but rather follows a non-standard distribution. Critical values are derived from Monte Carlo experiments in, for example, Fuller (1976). Relevant examples of the distribution are shown in table 4.1 below

## Critical Values for the DF Test

Significance level	10%	5%	1%
C.V. for constant but no trend	-2.57	-2.86	-3.43
C.V. for constant and trend	-3.12	-3.41	-3.96

Table 4.1: Critical Values for DF and ADF Tests (Fuller, 1976, p373).

The null hypothesis of a unit root is rejected in favour of the stationary alternative in each case if the test statistic is more negative than the critical value.

## The Augmented Dickey Fuller (ADF) Test

- ❖ The tests above are only valid if  $u_t$  is white noise. In particular,  $u_t$  will be autocorrelated if there was autocorrelation in the dependent variable of the regression ( $\Delta y_t$ ) which we have not modelled. The solution is to “augment” the test using  $p$  lags of the dependent variable. The alternative model in case (i) is now written:

$$\Delta y_t = \psi y_{t-1} + \sum_{i=1}^p \alpha_i \Delta y_{t-i} + u_t$$

- ❖ The same critical values from the DF tables are used as before. A problem now arises in determining the optimal number of lags of the dependent variable.

There are 2 ways

- use the frequency of the data to decide
- use information criteria

## Testing for Higher Orders of Integration

- ❖ Consider the simple regression:

$$\Delta y_t = \psi y_{t-1} + u_t$$

We test  $H_0: \psi=0$  vs.  $H_1: \psi<0$ .

- ❖ If  $H_0$  is rejected we simply conclude that  $y_t$  does not contain a unit root.
- ❖ But what do we conclude if  $H_0$  is not rejected? The series contains a unit root, but is that it? No! What if  $y_t \sim I(2)$ ? We would still not have rejected. So we now need to test

$$H_0: y_t \sim I(2) \text{ vs. } H_1: y_t \sim I(1)$$

We would continue to test for a further unit root until we rejected  $H_0$ .

- ❖ We now regress  $\Delta^2 y_t$  on  $\Delta y_{t-1}$  (plus lags of  $\Delta^2 y_t$  if necessary).
- ❖ Now we test  $H_0: \Delta y_t \sim I(1)$  which is equivalent to  $H_0: y_t \sim I(2)$ .
- ❖ So in this case, if we do not reject (unlikely), we conclude that  $y_t$  is at least  $I(2)$ .

## The Phillips-Perron Test

- ❖ Phillips and Perron have developed a more comprehensive theory of unit root nonstationarity. The tests are similar to ADF tests, but they incorporate an automatic correction to the DF procedure to allow for autocorrelated residuals.
- ❖ The tests usually give the same conclusions as the ADF tests, and the calculation of the test statistics is complex.

## Criticism of Dickey-Fuller and Phillips-Perron-type tests

- ❖ Main criticism is that the power of the tests is low if the process is stationary but with a root close to the non-stationary boundary.

e.g. the tests are poor at deciding if

$$\phi=1 \text{ or } \phi=0.95,$$

especially with small sample sizes.

- ❖ If the true data generating process (dgp) is

$$y_t = 0.95y_{t-1} + u_t$$

then the null hypothesis of a unit root should be rejected.

- ❖ One way to get around this is to use a stationarity test as well as the unit root tests we have looked at.

## Stationarity tests

- ❖ Stationarity tests have

$H_0: y_t$  is stationary

versus

$H_1: y_t$  is non-stationary

So that by default under the null the data will appear stationary.

- ❖ One such stationarity test is the KPSS test (Kwaitowski, Phillips, Schmidt and Shin, 1992).
- ❖ Thus we can compare the results of these tests with the ADF/PP procedure to see if we obtain the same conclusion.



## Stationarity tests (cont'd)

### ❖ A Comparison

ADF / PP

$H_0: y_t \sim I(1)$

$H_1: y_t \sim I(0)$

KPSS

$H_0: y_t \sim I(0)$

$H_1: y_t \sim I(1)$

### ❖ 4 possible outcomes

Reject  $H_0$

Do not reject  $H_0$

Reject  $H_0$

Do not reject  $H_0$

and

and

and

and

Do not reject  $H_0$

Reject  $H_0$

Reject  $H_0$

Do not reject  $H_0$

## Unit Root Tests with Structural Breaks

- ❖ The standard Dickey-Fuller-type unit root tests presented above do not perform well if there are structural breaks in the series
- ❖ The tests have low power in such circumstances and they fail to reject the unit root null hypothesis when it is incorrect as the slope parameter in the regression of  $y_t$  on  $y_{t-1}$  is biased towards unity
- ❖ The larger the break and the smaller the sample, the lower the power of the test
- ❖ Unit root tests are also oversized in the presence of structural breaks
- ❖ Perron (1989) demonstrates that after allowing for structural breaks in the tests, a whole raft of macroeconomic series may be stationary
- ❖ He argues that most economic time series are best characterised by broken trend stationary processes, i.e. a deterministic trend but with a structural break.

## The Perron (1989) Procedure - Background

- ❖ Perron (1989) proposes three test equations differing dependent on the type of break that is thought to be present:
  1. A ‘crash’ model that allows a break in the level (i.e. the intercept)
  2. A ‘changing growth’ model that allows for a break in the growth rate (i.e. the slope)
  3. A model that allows for both types of break to occur at the same time, changing both the intercept and the slope of the trend.
- ❖ Define the break point in the data as  $T_b$  and  $D_t$  is a dummy variable defined as

$$D_t = \begin{cases} 0 & \text{if } t < T_b \\ 1 & \text{if } t \geq T_b \end{cases}$$

## The Perron (1989) Procedure - Details

- ❖ The equation for the third (most general) version of the test is

$$\Delta y_t = \psi y_{t-1} + \mu + \alpha_1 D_t + \alpha_2 (t - T_b) D_t + \lambda t + \sum_{i=1}^p \alpha_i \Delta y_{t-i} + u$$

- ❖ For the crash only model, set  $\alpha_2 = 0$
- ❖ For the changing growth only model, set  $\alpha_1 = 0$
- ❖ In all three cases, there is a unit root with a structural break at  $T_b$  under the null hypothesis and a series that is a stationary process with a break under the alternative
- ❖ A limitation of this approach is that it assumes that the break date is known in advance
- ❖ It is possible, however, that the date will not be known and must be determined from the data.

## The Banerjee et al. (1992) and Zivot and Andrews (1992) Procedures - Background

- ❖ More seriously, Christiano (1992) has argued that the critical values employed with the test will presume the break date to be chosen exogenously
- ❖ But most researchers will select a break point based on an examination of the data
- ❖ Thus the asymptotic theory assumed will no longer hold
- ❖ Banerjee et al. (1992) and Zivot and Andrews (1992) introduce an approach to testing for unit roots in the presence of structural change that allows the break date to be selected endogenously.
- ❖ Their methods are based on recursive, rolling and sequential tests.

# The Banerjee et al. (1992) and Zivot and Andrews (1992) Procedures - Details

- ❖ For the recursive and rolling tests, Banerjee et al. propose four specifications.
  1. The standard Dickey-Fuller test on the whole sample
  2. The ADF test conducted repeatedly on the sub-samples and the minimal DF statistic is obtained
  3. The maximal DF statistic obtained from the sub-samples
  4. The difference between the maximal and minimal statistics
- ❖ For the sequential test, the whole sample is used each time with the following regression being run

$$\Delta y_t = \psi y_{t-1} + \mu + \alpha \tau_t(t_{used}) + \lambda t + \sum_{i=1}^p \alpha_i \Delta y_{t-i} + u_t$$

where  $t_{used} = T_b/T$ .

# The Banerjee et al. (1992) and Zivot and Andrews (1992) Procedures – Details 2

- ❖ The test is run repeatedly for different values of  $T_b$  over as much of the data as possible (a ‘trimmed sample’)
- ❖ This excludes the first few and the last few observations
- ❖ Clearly it is  $\tau_t(t_{used})$  allows for the break, which can either be in the level (where  $\tau_t(t_{used}) = 1$  if  $t > t_{used}$  and 0 otherwise); or the break can be in the deterministic trend (where  $\tau_t(t_{used}) = t - t_{used}$  if  $t > t_{used}$  and 0 otherwise)
- ❖ For each specification, a different set of critical values is required, and these can be found in Banerjee et al.

## Further Extensions

- ❖ Perron (1997) proposes an extension of the Perron (1989) technique but using a sequential procedure that estimates the test statistic allowing for a break at any point during the sample to be determined by the data
- ❖ This technique is very similar to that of Zivot and Andrews, except that his is more flexible since it allows for a break under both the null and alternative hypotheses
- ❖ A further extension would be to allow for more than one structural break in the series – for example, Lumsdaine and Papell (1997) enhance the Zivot and Andrews (1992) approach to allow for two structural breaks.



# Testing for Unit Roots with Structural Breaks

## Example: EuroSterling Interest Rates

- ❖ Brooks and Rew (2002) examine whether EuroSterling interest rates are best viewed as unit root process or not, allowing for the possibility of structural breaks in the series
- ❖ Failure to account for structural breaks (caused, for example, by changes in monetary policy or the removal of exchange rate controls) may lead to incorrect inferences regarding the validity or otherwise of the expectations hypothesis.
- ❖ Their sample covers the period 1 January 1981 to 1 September 1997
- ❖ They use the standard Dickey-Fuller test, the recursive and sequential tests of Banerjee et al. They also employ the rolling test, the Perron (1997) approach and several other techniques

# Testing for Unit Roots with Structural Breaks in EuroSterling Interest Rates – Results

Maturity	$t_{DF}$	Recursive statistics			Sequential statistics	
		$\hat{t}_{DF}^{max}$	$\hat{t}_{DF}^{min}$	$\hat{t}_{DF}^{diff}$	$\tilde{t}_{DF,trend}^{min}$	$\tilde{t}_{DF,mean}^{min}$
Short rate	-2.44	-1.33	-3.29	1.96	-2.99	-4.79
7-days	-1.95	-1.33	-3.19	1.86	-2.44	-5.65
1-month	-1.82	-1.07	-2.90	1.83	-2.32	-4.78
3-months	-1.80	-1.02	-2.75	1.73	-2.28	-4.02
6-months	-1.86	-1.00	-2.85	1.85	-2.28	-4.10
1-year	-1.97	-0.74	-2.88	2.14	-2.35	-4.55
Critical values	-3.13	-1.66	-3.88	3.21	-4.11	-4.58

*Notes:* Source: Brooks and Rew (2002), taken from tables 1, 4 and 5.

$\tilde{t}_{DF,trend}^{min}$  denotes the sequential test statistic allowing for a break in the trend, while  $\tilde{t}_{DF,mean}^{min}$  is the test statistic allowing for a break in the level.

The final row presents the 10% level critical values for each type of test obtained from Banerjee et al. (1992, p. 278, table 2).

## Testing for Unit Roots with Structural Breaks in EuroSterling Interest Rates – Conclusions

- ❖ The findings for the recursive tests are that the unit root null should not be rejected at the 10% level for any of the maturities examined
- ❖ For the sequential tests, the results are slightly more mixed with the break in trend model not rejecting the null hypothesis, while it is rejected for the short, 7-day and the 1-month rates when a structural break is allowed for in the mean
- ❖ The weight of evidence indicates that short term interest rates are best viewed as unit root processes that have a structural break in their level around the time of ‘Black Wednesday’ (16 September 1992) when the UK dropped out of the European Exchange Rate Mechanism
- ❖ The longer term rates, on the other hand, are  $I(1)$  processes with no breaks

## Seasonal Unit Roots

- ❖ It is possible that a series may contain seasonal unit roots, so that it requires seasonal differencing to induce stationarity
- ❖ We would use the notation  $I(d,D)$  to denote a series that is integrated of order  $d,D$  and requires differencing  $d$  times and seasonal differencing  $D$  times to obtain a stationary process
- ❖ Osborn (1990) develops a test for seasonal unit roots based on a natural extension of the Dickey-Fuller approach.
- ❖ However, Osborn also shows that only a small proportion of macroeconomic series exhibit seasonal unit roots; the majority have seasonal patterns that can better be characterised using dummy variables, which may explain why the concept of seasonal unit roots has not been widely adopted

## Cointegration: An Introduction

- ❖ In most cases, if we combine two variables which are  $I(1)$ , then the combination will also be  $I(1)$ .
- ❖ More generally, if we combine variables with differing orders of integration, the combination will have an order of integration equal to the largest. i.e.,

if  $X_{i,t} \sim I(d_i)$  for  $i = 1, 2, 3, \dots, k$

so we have  $k$  variables each integrated of order  $d_i$ .

Let

$$z_t = \sum_{i=1}^k \alpha_i X_{i,t} \quad (1)$$

Then  $z_t \sim I(\max d_i)$

## Linear Combinations of Non-stationary Variables

- ❖ Rearranging (1), we can write

$$X_{1,t} = \sum_{i=2}^k \beta_i X_{i,t} + z'_t$$

where  $\beta_i = -\frac{\alpha_i}{\alpha_1}$ ,  $z'_t = \frac{z_t}{\alpha_1}$ ,  $i = 2, \dots, k$

- ❖ This is just a regression equation.
- ❖ But the disturbances would have some very undesirable properties:  $z'_t$  is not stationary and is autocorrelated if all of the  $X_i$  are  $I(1)$ .
- ❖ We want to ensure that the disturbances are  $I(0)$ . Under what circumstances will this be the case?

## Definition of Cointegration (Engle & Granger, 1987)

- ❖ Let  $z_t$  be a  $k \times 1$  vector of variables, then the components of  $z_t$  are cointegrated of order  $(d,b)$  if
  - i) All components of  $z_t$  are  $I(d)$
  - ii) There is at least one vector of coefficients  $\alpha$  such that  $\alpha' z_t \sim I(d-b)$
- ❖ Many time series are non-stationary but “move together” over time.
- ❖ If variables are cointegrated, it means that a linear combination of them will be stationary.
- ❖ There may be up to  $r$  linearly independent cointegrating relationships (where  $r \leq k-1$ ), also known as cointegrating vectors.  $r$  is also known as the cointegrating rank of  $z_t$ .
- ❖ A cointegrating relationship may also be seen as a long term relationship.

# Cointegration and Equilibrium

- ❖ Examples of possible Cointegrating Relationships in finance:
  - spot and futures prices
  - ratio of relative prices and an exchange rate
  - equity prices and dividends
- ❖ Market forces arising from no arbitrage conditions should ensure an equilibrium relationship.
- ❖ No cointegration implies that series could wander apart without bound in the long run.



## Equilibrium Correction or Error Correction Models

- ❖ When the concept of non-stationarity was first considered, a usual response was to independently take the first differences of a series of  $I(1)$  variables.
- ❖ The problem with this approach is that pure first difference models have no long run solution.  
e.g. Consider  $y_t$  and  $x_t$  both  $I(1)$ .  
The model we may want to estimate is
 
$$\Delta y_t = \beta \Delta x_t + u_t$$
 But this collapses to nothing in the long run.
- ❖ The definition of the long run that we use is where
 
$$y_t = y_{t-1} = y; x_t = x_{t-1} = x.$$
- ❖ Hence all the difference terms will be zero, i.e.  $\Delta y_t = 0; \Delta x_t = 0$ .

## Specifying an ECM

- ❖ One way to get around this problem is to use both first difference and levels terms, e.g.

$$\Delta y_t = \beta_1 \Delta x_t + \beta_2 (y_{t-1} - \gamma x_{t-1}) + u_t \quad (2)$$

- ❖  $y_{t-1} - \gamma x_{t-1}$  is known as the error correction term.
- ❖ Providing that  $y_t$  and  $x_t$  are cointegrated with cointegrating coefficient  $\gamma$ , then  $(y_{t-1} - \gamma x_{t-1})$  will be  $I(0)$  even though the constituents are  $I(1)$ .
- ❖ We can thus validly use OLS on (2).
- ❖ The Granger representation theorem shows that any cointegrating relationship can be expressed as an equilibrium correction model.

## Testing for Cointegration in Regression

- ❖ The model for the equilibrium correction term can be generalised to include more than two variables:

$$y_t = \beta_1 + \beta_2 x_{2t} + \beta_3 x_{3t} + \dots + \beta_k x_{kt} + u_t \quad (3)$$

- ❖  $u_t$  should be  $I(0)$  if the variables  $y_t, x_{2t}, \dots, x_{kt}$  are cointegrated.
- ❖ So what we want to test is the residuals of equation (3) to see if they are non-stationary or stationary. We can use the DF / ADF test on  $u_t$ .

So we have the regression

with  $v_t \sim \text{iid}$ .

$$\Delta \hat{u}_t = \psi \hat{u}_{t-1} + v_t$$

- ❖ However, since this is a test on the residuals of an actual model,  $\hat{u}_t$ , then the critical values are changed.

## Testing for Cointegration in Regression: Conclusions

- ❖ Engle and Granger (1987) have tabulated a new set of critical values and hence the test is known as the Engle Granger (E.G.) test.
- ❖ We can also use the Durbin Watson test statistic or the Phillips Perron approach to test for non-stationarity of  $\hat{u}_t$ .

$$\hat{u}_t$$

- ❖ What are the null and alternative hypotheses for a test on the residuals of a potentially cointegrating regression?

$H_0$  : unit root in cointegrating regression's residuals

$H_1$  : residuals from cointegrating regression are stationary

# Methods of Parameter Estimation in Cointegrated Systems: The Engle-Granger Approach

❖ There are (at least) 3 methods we could use: Engle Granger, Engle and Yoo, and Johansen.

❖ The Engle Granger 2 Step Method

This is a single equation technique which is conducted as follows:

*Step 1:*

- Make sure that all the individual variables are I(1).
- Then estimate the cointegrating regression using OLS.
- Save the residuals of the cointegrating regression,  $\hat{u}_t$ .
- Test these residuals to ensure that they are I(0).

*Step 2:*

- Use the step 1 residuals as one variable in the error correction model e.g.

$$\Delta y_t = \beta_1 \Delta x_t + \beta_2 (\hat{u}_{t-1}) + u_t$$

where  $\hat{u}_{t-1} = y_{t-1} - x_{t-1}$

$$\hat{u}_{t-1} = \hat{\gamma} \hat{u}_{t-1}$$

## An Example of a Model for Non-stationary Variables: Lead-Lag Relationships between Spot and Futures Prices

### Background

- ❖ We expect changes in the spot price of a financial asset and its corresponding futures price to be perfectly contemporaneously correlated and not to be cross-autocorrelated.

i.e. expect

$$\begin{aligned}\text{Corr}(\Delta \ln(F_t), \Delta \ln(S_t)) &\approx 1 \\ \text{Corr}(\Delta \ln(F_t), \Delta \ln(S_{t-k})) &\approx 0 \quad \forall k \\ \text{Corr}(\Delta \ln(F_{t-j}), \Delta \ln(S_t)) &\approx 0 \quad \forall j\end{aligned}$$

- ❖ We can test this idea by modelling the lead-lag relationship between the two.
- ❖ We will consider two papers Tse(1995) and Brooks *et al* (2001).

## Futures & Spot Data

- ❖ Tse (1995): 1055 daily observations on NSA stock index and stock index futures values from December 1988 - April 1993.
- ❖ Brooks *et al* (2001): 13,035 10-minutely observations on the FTSE 100 stock index and stock index futures prices for all trading days in the period June 1996 – 1997.

## Methodology

- ❖ The fair futures price is given by

where  $F_t^*$  is the fair futures price,  $S_t$  is the spot price,  $r$  is a continuously compounded risk-free rate of interest,  $d$  is the continuously compounded yield in terms of dividends derived from the stock index until the futures contract matures, and  $(T-t)$  is the time to maturity of the futures contract. Taking logarithms of both sides of equation above gives

- ❖ First, test  $f_t$  and  $s_t$  for nonstationarity.

$$f_t^* = s_t + (r - d)(T - t)$$



## Dickey-Fuller Tests on Log-Prices and Returns for High Frequency FTSE Data

	Futures	Spot
Dickey-Fuller Statistics for Log-Price Data	-0.1329	-0.7335
Dickey Fuller Statistics for Returns Data	-84.9968	-114.1803

## Cointegration Test Regression and Test on Residuals

- ❖ Conclusion:  $\log F_t$  and  $\log S_t$  are not stationary, but  $\Delta \log F_t$  and  $\Delta \log S_t$  are stationary.
- ❖ But a model containing only first differences has no long run relationship.
- ❖ Solution is to see if there exists a cointegrating relationship between  $f_t$  and  $s_t$  which would mean that we can validly include levels terms in this framework.
- ❖ Potential cointegrating regression:

$$s_t = \gamma_0 + \gamma_1 f_t + z_t$$

where  $z_t$  is a disturbance term.

- ❖ Estimate the regression, collect the residuals,  $\hat{z}_t$ , and test whether they are stationary.

## Estimated Equation and Test for Cointegration for High Frequency FTSE Data

Cointegrating Regression	
Coefficient	Estimated Value
$\hat{\gamma}_0$	0.1345
$\hat{\gamma}_1$	0.9834
DF Test on residuals	Test Statistic
$\hat{z}_t$	-14.7303

## Conclusions from Unit Root and Cointegration Tests

- ❖ Conclusion: are stationary and therefore we have a cointegrating relationship between  $\log F_t$  and  $\log S_t$ .

$$\hat{\tilde{z}}_t$$

- ❖ Final stage in Engle-Granger 2-step method is to use the first stage residuals, as the equilibrium correction term in the general equation.

- ❖ The overall model is

$$\Delta \ln S_t = \beta_0 + \delta \hat{\tilde{z}}_{t-1} + \beta_1 \Delta \ln S_{t-1} + \alpha_1 \Delta \ln F_{t-1} + v_t$$

❖ Cek file pdf slide Robert Nau mengenai permodelan ARIMA

TERIMA KASIH